ONCE MORE ON POSITIVE COMMUTATORS

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ABSTRACT. Let A and B be bounded operators on a Banach lattice E such that the commutator $C = AB - BA$ and the product BA are positive operators. If the product AB is a power-compact operator, then C is a quasi-nilpotent operator having a triangularizing chain of closed ideals of E. This theorem answers an open question posed in [3], where the study of positive commutators of positive operators has been initiated.

1. INTRODUCTION

Let X be a Banach space. The spectrum and the spectral radius of a bounded operator T on X are denoted by $\sigma(T)$ and $r(T)$, respectively. A bounded operator T on X is said to be *power-compact* if T^n is a compact operator for some $n \in \mathbb{N}$. A *chain* $\mathcal{C}$ is a family of closed subspaces of X that is totally ordered by inclusion. We say that $\mathcal{C}$ is a *complete* chain if it is closed under arbitrary intersections and closed linear spans. If $\mathcal{M}$ is in a complete chain $\mathcal{C}$, then the *predecessor* $\mathcal{M}_-$ of $\mathcal{M}$ in $\mathcal{C}$ is defined as the closed linear span of all proper subspaces of $\mathcal{M}$ belonging to $\mathcal{C}$.

Let E be a Banach lattice. An operator T on E is called *positive* if the positive cone E^+ is invariant under T. It is well-known that every positive operator T is bounded and that $r(T)$ belongs to $\sigma(T)$. A bounded operator T on E is said to be *ideal-reducible* if there exists a non-trivial closed ideal of E invariant under T. Otherwise, it is *ideal-irreducible*. If the chain $\mathcal{C}$ of closed ideals of E is maximal in the lattice of all closed ideals of E and if every one of its members is invariant under an operator T on E, then $\mathcal{C}$ is called a *triangularizing chain* for T, and T is said to be *ideal-triangularizable*. Note that such a chain is also maximal in the lattice of all closed subspaces of E (see e.g. [4, Proposition 1.2]).

In [3] positive commutators of positive operators on Banach lattices are studied. The main result [3, Theorem 2.2] is the following

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Theorem 1.1. *Let A and B be positive compact operators on a Banach lattice E such that the commutator $C = AB - BA$ is also positive. Then C is an ideal-triangularizable quasi-nilpotent operator.*

Examples in [3] show that the compactness assumption of Theorem 1.1 cannot be omitted. They are based on a simple example that can be obtained by setting $A = S^*$ and $B = S$, where S is the unilateral shift on the Banach lattice l^2.

Theorem 1.1 has been futher extended in [5, Theorem 3.4]. Recall that a bounded operator T on a Banach space is called a *Riesz operator* or an *essentially quasi-nilpotent operator* if $\{0\}$ is the essential spectrum of T.

Theorem 1.2. *Let A and B be positive operators on a Banach lattice E such that the sum $A+B$ is a Riesz operator. If the commutator $C = AB - BA$ is a power-compact positive operator, then it is an ideal-triangularizable quasi-nilpotent operator.*

In this note we answer affirmatively the open question posed in [3, Open questions 3.7 (1)] whether is it enough to assume in Theorem 1.1 that only one of the operators A and B is compact.

2. Preliminaries

If T is a power-compact operator on a Banach space X, then, by the classical spectral theory, for each $\lambda \in \mathbb{C} \setminus \{0\}$ the operator $\lambda - T$ has finite ascent k, i.e., k is the smallest natural number such that $\ker((\lambda - T)^k) = \ker((\lambda - T)^{k+1})$. In this case the *(algebraic) multiplicity* $m(T, \lambda)$ of λ is the dimension of the subspace $\ker((\lambda - T)^k)$.

We will make use of the following extension of Ringrose's Theorem.

Theorem 2.1. *Let T be a power-compact operator on a Banach space X, and let $\mathcal{C}$ be a complete chain of closed subspaces invariant under T. Let $\mathcal{C}'$ be a subchain of $\mathcal{C}$ of all subspaces $\mathcal{M} \in \mathcal{C}$ such that $\mathcal{M}_- \neq \mathcal{M}$. For each $\mathcal{M} \in \mathcal{C}'$, define $T_{\mathcal{M}}$ to be the quotient operator on $\mathcal{M}/\mathcal{M}_-$ induced by T. Then*

$$\sigma(T) \setminus \{0\} = \bigcup_{\mathcal{M} \in \mathcal{C}'} \sigma(T_{\mathcal{M}}) \setminus \{0\}.$$

Moreover, for each $\lambda \in \mathbb{C} \setminus \{0\}$ we have

$$m(T, \lambda) = \sum_{\mathcal{M} \in \mathcal{C}'} m(T_{\mathcal{M}}, \lambda).$$

Proof. In the case of a compact operator T the first equality is proved in [12, Theorem 7.2.7], while the second equality follows from the theorem

[12, Theorem 7.2.9] asserting that the algebraic multiplicity of each nonzero eigenvalue of T is equal to its diagonal multiplicity with respect to any triangularizing chain.

An inspection of the proofs of these theorems reveals that it is enough to assume that the operator T is power-compact. Moreover, in [8] the first equality was extended even to the case of polynomially compact operators. □

We will also need Pietsch's principle of related operators (see [11, 3.3.3]).

Theorem 2.2. *Let A and B be bounded operators on a Banach space. If AB is power-compact, then BA is power-compact and*

$$m(AB, \lambda) = m(BA, \lambda)$$

for each $\lambda \in \mathbb{C} \setminus \{0\}$.

The following theorem is a consequence of [9, Theorem 4.3]; see a recent paper [7, Theorem 0.1] which also contains the easily proved proposition [7, Proposition 0.2] that a positive operator is ideal-irreducible if and only if it is semi non-supporting (the notion used in [9]).

Theorem 2.3. *Let S and T be positive operators on a Banach lattice E such that $S \leq T$ and $r(S) = r(T)$. If T is an ideal-irreducible power-compact operator, then $S = T$.*

3. Results

The main result of this note is the following extension of Theorem 1.1 (and [3, Theorem 2.4] as well).

Theorem 3.1. *Let A and B be bounded operators on a Banach lattice E such that $AB \geq BA \geq 0$ and AB is a power-compact operator. Then the commutator $C = AB - BA$ is an ideal-triangularizable quasi-nilpotent operator.*

Proof. Let $\mathcal{C}$ be a chain (of closed ideals) that is maximal in the lattice of all closed ideals invariant under AB. By maximality, this chain is complete. Let $\mathcal{C}'$ be a subchain of all subspaces $\mathcal{M} \in \mathcal{C}$ such that $\mathcal{M}_- \neq \mathcal{M}$. Since $AB \geq BA \geq 0$ and $AB \geq C \geq 0$, every member of $\mathcal{C}$ is also invariant under the operators BA and C, and these operators are power-compact operators by the Aliprantis-Burkinshaw theorem [2, Theorem 5.14]. For any ideal $\mathcal{M} \in \mathcal{C}'$, $r((AB)_{\mathcal{M}}) \geq r((BA)_{\mathcal{M}})$, since $(AB)_{\mathcal{M}} \geq (BA)_{\mathcal{M}} \geq 0$. We will prove that $r((AB)_{\mathcal{M}}) = r((BA)_{\mathcal{M}})$ for every ideal $\mathcal{M} \in \mathcal{C}'$, and so $(AB)_{\mathcal{M}} = (BA)_{\mathcal{M}}$ by Theorem 2.3.

Assume there are ideals $\mathcal{M} \in \mathcal{C}'$ such that $r((AB)_{\mathcal{M}}) > r((BA)_{\mathcal{M}})$. Among them choose $\mathcal{M}_0 \in \mathcal{C}'$ for which $\lambda_0 := r((AB)_{\mathcal{M}_0})$ is maximal. Such an ideal exists, because for each $\epsilon > 0$ there are only finitely many eigenvalues of AB with the absolute value at least ϵ. For each ideal $\mathcal{M} \in \mathcal{C}'$ with $r((AB)_{\mathcal{M}}) > \lambda_0$, we must have $r((AB)_{\mathcal{M}}) = r((BA)_{\mathcal{M}})$, and so $(AB)_{\mathcal{M}} = (BA)_{\mathcal{M}}$ by Theorem 2.3. The same conclusion holds in the case when $r((AB)_{\mathcal{M}}) = r((BA)_{\mathcal{M}}) = \lambda_0$. If $\lambda_0 = r((AB)_{\mathcal{M}}) > r((BA)_{\mathcal{M}})$, then

$$m((AB)_{\mathcal{M}}, \lambda_0) > 0 = m((BA)_{\mathcal{M}}, \lambda_0).$$

If $r((AB)_{\mathcal{M}}) < \lambda_0$, then

$$m((AB)_{\mathcal{M}}, \lambda_0) = 0 = m((BA)_{\mathcal{M}}, \lambda_0).$$

In view of Theorem 2.1 we now conclude that $m(AB, \lambda_0) > m(BA, \lambda_0)$. However, by Theorem 2.2, we have $m(AB, \lambda_0) = m(BA, \lambda_0)$. This contradiction shows that, for each $\mathcal{M} \in \mathcal{C}'$, $(AB)_{\mathcal{M}} = (BA)_{\mathcal{M}}$ and so $C_{\mathcal{M}} = (AB)_{\mathcal{M}} - (BA)_{\mathcal{M}} = 0$. By Theorem 2.1, we conclude that C is quasi-nilpotent.

Finally, it is a simple consequence (see e.g. [5, Theorem 1.3]) of the well-known de Pagter's theorem (see [1, Theorem 9.19] or [10]) that C has a triangularizing chain of closed ideals of E. In fact, we can simply complete the chain $\mathcal{C}$ to a triangularizing chain of closed ideals for the operator C. $\square$

As a corollary we obtain the answer to an open question posed in [3, Open questions 3.7 (1)].

Corollary 3.2. *Let A and B be positive operators on a Banach lattice E such that the commutator $C = AB - BA$ is a positive operator. If one of the operators A and B is power-compact (in particular, compact), then the commutator C is an ideal-triangularizable quasi-nilpotent operator.*

Proof. By a simple induction, we have $0 \leq (AB)^n \leq A^n B^n$ for every $n \in \mathbb{N}$. Assume now that for $n \in \mathbb{N}$ one of the operators A^n and B^n is compact, so that the operator $A^n B^n$ is compact. Then the operator $(AB)^{3n}$ is also compact by the Aliprantis-Burkinshaw theorem [2, Theorem 5.14]. Therefore, Theorem 3.1 can be applied. $\square$

It should be noted that a recent preprint [6, Theorem 4.5] gives an independent proof of Corollary 3.2 in the case when one of the operators A and B is compact.

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